Multiple State Models

Two State Models

The first half of the CT5 course considers the two state model:

ALIVE and DEAD

This leads us to some basic actuarial relationships such as

$$ {}_tp_x = \frac{l_{x+t}}{l_x} $$

$$ {}_tq_x = 1 -_t p_x $$

$$ {}_tp_x = e^{-\int_0^t \mu_{x+s} ds} $$

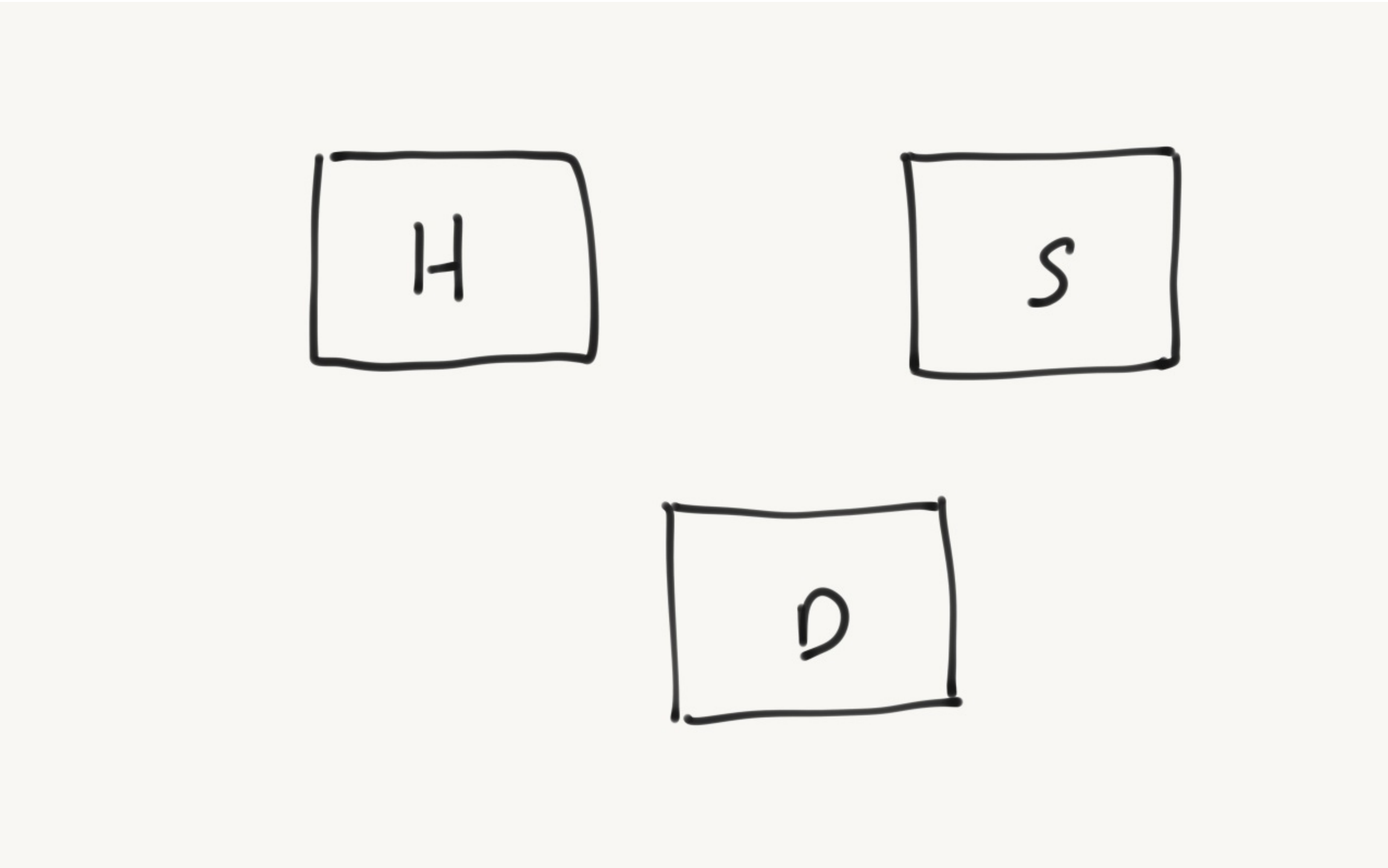
In the second half of the course we widen this model to consider other states

This poses 2 fundamental questions:

- What happens to state probabilities if we can return to a state
- What happens to transition probabilities when there are multiple decrements interacting

Returning to a State

The ***Healthy Sick Dead*** model is the classic example of the mathematical problems that arise if you can return to states you have left



0 1 2 3 4

What can you do about this mathematically

Can these be solved

This is covered in subject CT4 are we do not need to consider it further here

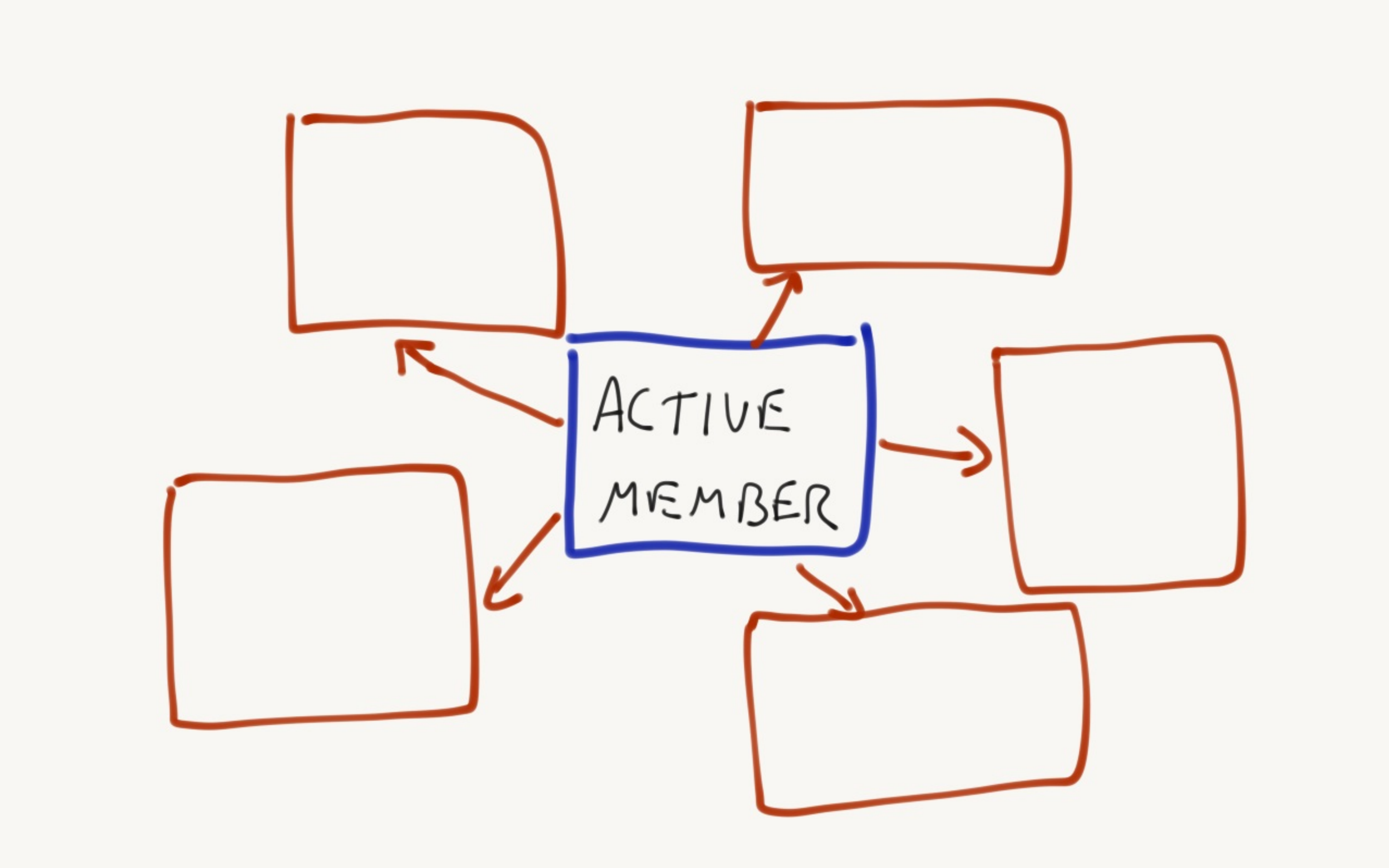
Multiple Decrements

The other problem (which does concern us here) is where many different decrements can happen from the same state

Examples of this could be

0 1 2 3 4

What kind of decrement might members of pension schemes experience



0 1 2 3 4 5

Illustrative Question

If there are 1000 members of a pension scheme aged 50 at the start of the year and 100 leave during the year and you know that ${}_1q_{50} = 0.02$, how many members do you expect to die during the year?

Would you estimate

20:

19:

18:

unknown:

No vote

What do you think the force of mortality is given some members are leaving

-ln(0.98):

less than -ln(0.98):

more than -ln(0.98):

unknown:

No vote

We need to think about μ

Remember: $\mu_x = \lim_{h \to 0} \frac{{}_hq_x}{h}$

? *Given the definition of μ how do you believe multiple decrements interact with each other when considered in terms of μ*

Or using actuarial notation:

for any decrements d.

The left hand side μ_x^d refers to the underlying single decrement rate and the right hand side $(a\mu)_x^d$ refers to the actual experienced decrement (from decrement d) given the existance of the other decrements

What about ${}_tp_x$ and ${}_tq_x$

? *Is it true to say: ${}_tq_x^d = (aq)_x^d$ given the same extension to actuarial notation of the $'a'$ referring to the actual decrement probability in the context of the other decrements*

? *Why*

Notation

We should probably take a look at multiple decrement notation at this point:

${}_tp_x^{aa}$ is the probability of being in state a at age $x+t$ having been in state a at age x

a may refer to alive here but it may just be the first decrement out of a, b, c etc

${}_tp_x^{ad}$ is the probability of being in state d at age $x+t$ having been in state a at age x

${}_tp_x^{\overline{aa}}$ is the probability of being in state a at age $x+t$ having been in state a at age x and never leaving state a between age x and age $x+t$

You may notice that once you have more than 2 states q starts to lose its significance, although where, for example, you have one alive state and multiple types of death then you can still see it:

${}_tq_x^{d_1}$ for example would be the probability of dying from cause 1 between age x and age $x+t$

When you introduce an a before the p or q or μ then instead of being the underlying vaue it becomes the experienced value in the context of other decrements

So while ${}_tp_x^{ad}$ shows the underlying probability of moving from a to d ignoring other decrements ${}_t(ap)_x^{ad}$ shows this probability adjusted for the fact that other decrements are taking place

The logic of this notation then extends in a perfectly sensible way. For example

$(ad)_x^{d_1}$ is the actual number of deaths experienced through cause 1 for a life aged x over a year.

Complicated? or not?

The thing to remember is:

$$\mu_x^d = (a\mu)_x^d$$

So the process must always be: Get back to the μ and solve everything from there.

Question:

? *If there are two decrements: death(d) and retirement (r) and it is known that their respective forces of decrement are: μ_x^d and μ_x^r what is their combined force of decrement?*

? *What is the probability that an active member of a pension scheme aged x is still an active member in t years time*

Illustrative example

Thinking back to the example we started with.

A pension scheme has 1000 members aged 50 and over the year 100 leave the scheme. The mortality rate for these members is given by ${}_1q_{50} = 0.02$.

How many members (aged 50) do you expect to die in the year, assuming that both leaving and death experience a constant force of decrement?

Difficulty: the 100 leavers is a 'multiple decrement' figure as this is the number of leavers in the context of a population which is being reduced by death.

Easy part: $1 - q^d_{50} = e^{-\int_0^t \mu^d dt} \therefore \mu^d = -ln(0.98) = 0.0202$

We now have two more equations to use:

Total decrements $= 1000 - 1000 \times e^{-\mu^r - \mu^d}$

And what else can we say about these two decrements:

Why

so

Given that $(ad)^r_{50} + (ad)^d_{50}$ IS the total decrements then we have two equations in two unknowns:

$$100 + (ad)^d_{50} = 1000 - 1000 \times e^{-\mu^r - 0.0202} \text{....(1)}$$

and

$$\frac{100}{100 + (ad)^d_{50}} = \frac{\mu^r}{\mu^r + 0.0202} \text{....(2)}$$

If we substitute (1) into (2) we get:

$$\frac{100}{1000-1000\times e^{-\mu^r-0.0202}} = \frac{\mu^r}{\mu^r+0.0202}$$

Clearly this requires a numerical approach:

so let $f(\mu^r) = \frac{100}{1000-1000\times e^{-\mu^r-0.0202}} - \frac{\mu^r}{\mu^r+0.0202}$

$f(0.1) = 0.051001$

$f(0.2) = -0.4023$

$f(0.111251) = -0.0345$

$f(0.102927) = 0.027251$

$f(0.106601) = -0.00100$

$f(0.106471) = -0.000028$

so $\mu^r = 0.10647$

and $(ad)^d_{50} = 900 - 1000 \times e^{-0.10647-0.0202} = 18.97$

Note: the reason why we needed a numerical solution for this question was that we were given both a multiple decrement value (the 100) and a single decrement value (0.02). many questions are actually easier:

Supposing the 100 value you were given was from a single decemrent table where the impact of other decrements had already been backed out. How would this effect the question

What kind of technique might have been used to produce this table

Exam Style Question

The table below gives the causes of death for a sample population of elderly people over one year periods:

Age	population	cancer	heart disease	other
90	100000	2500	3200	6500
91	87800	2700	3900	5400
92	75800	3000	4000	6000
93	62800	nd	nd	nd

i) Assuming that each decrement has a uniform force of decrement over the year how many deaths would you expect from heart disease for people aged 90 if the force of mortality for cancer was halved for people aged 90.

Strategy: work out the μs and then calculate the probabilities

and

If we halve the force of mortality for cancer this becomes 0.01333

We then reconstruct the table so now the total force of mortality $= 0.01333 + 0.03413 + 0.06932 = 0.11678$

so the total deaths will be

These deaths will be in the ratio of the forces of mortality so

ii) If the forces of mortality remain the same as now for age 91, how many heart disease deaths would we now expect for the age 91 population.

Trick: you may think you need to go through the same process here but you do not.

:

Central rates of decrement

These are essentially a weighted average of the force of decrement over the given time interval:

Look at the definition (for all causes):

$$(am)_x = \frac{\int_0^1 (al)_{x+t}(a\mu)_{x+t}dt}{\int_0^1 (al)_{x+t}dt} = \frac{(ad)_x}{(aL)_x}$$

and just for cause k

$$(am)_x^k = \frac{\int_0^1 (al)_{x+t}(a\mu)_{x+t}^k dt}{\int_0^1 (al)_{x+t}dt} = \frac{(ad)_x^k}{(aL)_x}$$

Can you explain: $\int_0^1 (al)_{x+t}(a\mu)_{x+t}dt = (ad)_x$

Can you explain: $\int_0^1 (al)_{x+t}dt = (aL)_x$

? *Would it be true to say that* $(am)_x = \sum_{k=1}^{m}(am)_x^k$

? *Is the central rate of mortality an important concept*

Exam Style Question (easy)

A population of bees experiences two decrements (death and leaving the hive). The force of decrement from death is 0.2 and the force of decrement from leaving the hive is 0.1 for the first 4 months of the year and 0.4 for the last 8 months of the year. Out of a population of 2000 bees how many do you expect to die during the year.

You need to perform a 2 stage anlalysis here:

First 4 months

Population after 4 months = $2000 \times e^{-\frac{1}{3}(\mu_d+\mu_l)} = 2000e^{-0.1} = 1810$

So total decrements is 190 in the ratio of 0.2 deaths to 0.1 leavers so there are $\frac{2}{3} \times 190 = 127$ deaths

For second stage

Population at end = $1810 \times e^{-\frac{2}{3}\times(0.2+0.4)} = 1213$

So a further 597 decrements have occured of which $\frac{0.2}{0.2+0.4} = 199$ are deaths so

The total number of deaths is $127 + 199 = 326$

Exam Style Question (hard)

In a workforce of 1000 people there are 2 decrements: leave to a competitor and leave the industry. In the last year 200 people left for a competitor and 120 left the industry. However the rate (force of decrement) at which people left the industry doubled half way through the last year. Assuming all forces of decrement are constant other than where stated, derive a system of equations which could be solved to calculate the two forces of decrement.

Let μ_c be the rate at which people leave for a competitor

Let μ_i be the rate people currently leave the industry

Let x_c be leavers to competitor in first half of year and x_i analagously

Then $\frac{x_c}{x_c+x_i} = \frac{\mu_c}{\mu_c+0.5\mu_i}$

and $\frac{200-x_c}{200-x_c+120-x_i} = \frac{\mu_c}{\mu_c+\mu_i}$

and $\frac{1000-x_c-x_i}{1000} = e^{-\int_0^{0.5} \mu_c+0.5\mu_i dt} = e^{-0.5\mu_c-0.25\mu_i}$

and $\frac{680}{1000-x_c-x_i} = e^{-\int_{0.5}^{1} \mu_c+\mu_i dt} = e^{-0.5\mu_c-0.5\mu_i}$

Exercise

Develop a spreadsheet to solve this system of equations

Hint: You need to ask yourself the question: How many unknowns there really are

Solution You can find a spreadsheet here